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ON THE FULLY COMMUTATIVE ELEMENTS OF TYPE $\tilde{C}$ AND FAITHFULNESS OF RELATED TOWERS

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ABSTRACT. We define a tower of injections of $\tilde{C}$-type Coxeter groups $W(\tilde{C}_n)$ for $n \geq 1$. We define a tower of Hecke algebras and we use the faithfulness at the Coxeter level to show that this last tower is a tower of injections. Let $W^c(\tilde{C}_n)$ be the set of fully commutative elements in $W(\tilde{C}_n)$, we classify the elements of $W^c(\tilde{C}_n)$ and give a normal form for them. We use this normal form to define two injections from $W^c(\tilde{C}_{n-1})$ into $W^c(\tilde{C}_n)$. We then define the tower of affine Temperley-Lieb algebras of type $\tilde{C}$ and use the injections above to prove the faithfullness of this tower.

Affine Coxeter groups; affine Hecke algebra; affine Temperley-Lieb algebra; fully commutative elements.

1. INTRODUCTION

In his fundamental paper [15], in 1987, Jones has shown that the original Temperley-Lieb algebra, a certain finite-dimensional algebra appearing in statistical mechanics as well as in physics, could be viewed as a quotient of the Hecke algebra of type A_n [15, §11 and Note 13.20]. He also showed that the Markov trace, the famous invariant of braids, was actually a trace defined on the tower of Temperley-Lieb algebras of type A — the natural embedding of a Coxeter diagram of type A_n into a Coxeter diagram of type A_{n+1}, for $n \geq 1$, inducing monomorphisms of the associated Coxeter groups, Hecke algebras and Temperley-Lieb algebras.

This opened a door to intensive study. In the same year (1995), Fan [9] (in the simply laced case) and Graham [10], in their theses, defined and studied generalized Temperley-Lieb algebras, that are quotients of Hecke algebras of Coxeter groups generalizing the original one, and identified a basis for them, indexed by elements of the associated Coxeter group enjoying special properties, while Stembridge [16] defined and studied *fully commutative elements* in a Coxeter group:

An element w of a Coxeter group W is fully commutative *if any reduced expression for w can be obtained from any other by means of braid relations that only involve commuting generators.*

Since then, fully commutative elements of Coxeter groups, that do index a basis of the associated generalized Temperley-Lieb algebra, have been studied in their

own right by numerous authors, as well as generalized Temperley-Lieb algebras. Nevertheless the case of affine Coxeter groups was not much studied until recently.

An obvious difficulty of the affine case is that we have to deal with infinite groups and infinite-dimensional algebras. Yet a main difficulty of the affine case is the lack of parabolicity. In the series of finite Coxeter groups $(W(A_n))_{n\geq 1}$, $W(B_n)_{n\geq 2}$, $W(D_n)_{n\geq 3}$, the n-th group is a parabolic subgroup of the $(n+1)$-th one, so that the associated Hecke algebras and Temperley-Lieb algebras inject naturally into one another (that is, the n-th into the $(n+1)$-th). This is no longer the case for affine Coxeter groups, for which defining morphisms and proving injectivity is not straightforward.

For instance, the author in his thesis [1] defined and studied a tower of affine Temperley-Lieb algebras of type $\tilde{A}$ and defined on this tower the notion of a Markov trace, for which he proved existence and uniqueness, hence giving an invariant of affine links [2]. A crucial tool in this study was to produce a normal form for fully commutative elements in Coxeter groups of type $\tilde{A}$. It is only later that he could prove the faithfulness of this tower, by means of a faithful tower of fully commutative elements of type $\tilde{A}$ depending on their normal form [3].

The center of interest of the present work is this "tower"-point of view on the structures of type $\tilde{C}$, namely Coxeter groups, Hecke algebras and Temperley-Lieb algebras, for which fully commutative elements index at least two well-known bases. The heart of this work is the production of a normal form for fully commutative elements in Coxeter groups of type $\tilde{C}$, normal form that is subsequently used to build an injective tower of fully commutative elements of type $\tilde{C}$ and ultimately prove the faithfulness of the tower of Temperley-Lieb algebras of type $\tilde{C}$ that we define.

In his thesis [7] Ernst has given a faithful diagrammatic presentation for $\tilde{C}$-type Temperley-Lieb algebras (see also [8]). The method there was to classify fully commutative elements in Coxeter groups of type $\tilde{C}$ that are irreducible under "weak star reduction". In this paper we use simple algebraic methods, in particular the notion of affine length (Definition 4.4) to classify fully commutative elements by giving a normal form for each (Theorem 4.7).

This is the second affine normal form of fully commutative elements, after the one for type $\tilde{A}$ in [3]. The author has also obtained such a normal form in types $\tilde{B}$ and $\tilde{D}$, these normal forms allowed us in an unpublished work to calculate the length generating function $\sum_{w\in \tilde{W}^c} q^{l(w)}$ explicitly, i.e. as a rational polynomial in q (see [13]). We notice that affine fully commutative elements do not behave in a wild way in the four infinite families of affine Coxeter groups: they have a subword that is a power of a Coxeter element, except for the $\tilde{C}$ type where we have to distinguish between two families of elements, one of which involves as a subword a power of the

element $[-n, n]$ $t_{n+1} = \sigma_n\sigma_{n-1}\ldots\sigma_1 t\sigma_1\ldots\sigma_{n-1}\sigma_n t_{n+1}$ (Theorem 4.7). For normal forms of fully commutative elements in the finite Coxeter groups of types A, B and D we refer to [17].

Authors in [4] have studied cyclically fully commutative elements. We give in Remark 4.9 some examples of the way in which our normal form can express such elements.

Another motive for learning more about fully commutative elements of type $\tilde{C}$ is to give an answer to Green's hypothesis, see [12], especially Property B, that is the existence of a symmetric bilinear form with some nice properties linked to a known method given by Green (see for example [11]) about a Jones-like trace allowing us to compute the coefficients of Kazhdan-Lusztig polynomials. In the $\tilde{A}$ case for example the author has used the normal form to define a class of elements such that any trace is uniquely defined by its value on them [2, Definition 4.5, Theorem 4.6], and that is what we intend to do later with the type $\tilde{C}$: we wish to define and classify such traces in the type $\tilde{C}$. The topological interest here comes from the fact that the closures of $\tilde{C}$-type braids are the links in a double torus.

A further motivation is linked to the "parabolic-like presentation" defined in [1] in the $\tilde{A}$ case and recently by the author for type $\tilde{C}$. We would like to prove that the behavior of our structures does not change much if we define them with another presentation, where the tower would be defined by adding a generator, at the cost of replacing a braid relation by a "braid-like" relation.

This paper is divided into three parts and it is organized as follows:

The first part is centered around the towers of Coxeter groups and Hecke algebras of type $\tilde{C}$. In section 2, we define a group morphism from the Coxeter group $W(\tilde{C}_{n-1})$ of type $\tilde{C}_{n-1}$ to $W(\tilde{C}_n)$ and we prove the injectivity of this morphism by considering $W(\tilde{C}_{n-1})$ and $W(\tilde{C}_n)$ as subgroups of $W(\tilde{A}_{2n-1})$ and $W(\tilde{A}_{2n+1})$ respectively (Corollary 2.2). In section 3, we let K be a commutative ring with identity and we let q be an invertible element of K. We define the tower of $\tilde{C}$-type Hecke algebras $H\tilde{C}_n(q)$. We prove the injectivity of this tower for $K = \mathbf{Q}[q, q^{-1}]$ using the specialization $q = 1$ (Proposition 3.3).

The second part, section 4, is centered around the normal form of fully commutative elements of $W(\tilde{C}_n)$. After recalling the normal form in type B given by Stembridge in [17, Theorem 5.1], we define the affine length of an element of $W(\tilde{C}_n)$ and we establish the $\tilde{C}$-version of Stembridge's result, namely Theorem 4.7, that determines a normal form for fully commutative elements of $W(\tilde{C}_n)$. This is the main result in this section and the base point of what follows.

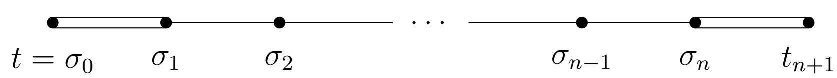
In the third part we describe two towers of fully commutative elements which will lead to the faithfulness of the tower of $\tilde{C}$-type Temperley-Lieb algebras. In section 5, we define two injections I and J from the set $W^c(\tilde{C}_{n-1})$ of fully commutative elements in $W(\tilde{C}_{n-1})$ into $W^c(\tilde{C}_n)$ and their essential properties, this is Theorem 5.2, of which the proof depends totally on the normal form. In section 6, we define the tower of $\tilde{C}$-type Temperley-Lieb algebras, then, as an application of our normal form, we prove the faithfulness of the arrows of this tower in Theorem 6.4. The proof uses in a crucial way the injections I and J of the previous section.

2. A faithful tower of $\tilde{C}$-type Coxeter groups

Let (W, S) be a Coxeter system with associated Coxeter diagram Γ. Let $w \in W(\Gamma)$ or simply W. We denote by $l(w)$ the length of a (any) reduced expression of w. We define $\mathscr{L}(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other terms s appears at the left edge of some reduced expression of w. We define $\mathscr{R}(w)$ similarly, on the right.

Consider the B-type Coxeter group with $n+1$ generators $W(B_{n+1})$, with the following Coxeter diagram:

Now let $W(\tilde{C}_{n+1})$ be the affine Coxeter group of $\tilde{C}$-type with $n+2$ generators in which $W(B_{n+1})$ could be seen a parabolic subgroup in two ways. We make our choice by presenting $W(\tilde{C}_{n+1})$ with the following Coxeter diagram:



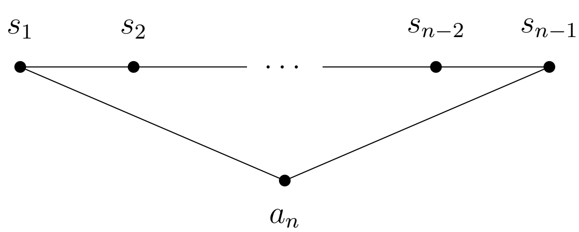
In other words the group $W(\tilde{C}_{n+1})$ has a presentation given by the set of generators $\{\sigma_0, \sigma_1, \ldots, \sigma_n, t_{n+1}\}$ and the relations:

$$
\begin{aligned}
&t_{n+1}^2 = 1 \text{ and } \sigma_i^2 = 1 \text{ for } 0 \le i \le n;\\
&\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } 0 \le i, j \le n,\ |i-j| \ge 2;\\
&\sigma_i t_{n+1} = t_{n+1}\sigma_i \text{ for } 0 \le i < n;\\
&\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-1;\\
&\sigma_0\sigma_1\sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_1\sigma_0;\\
&\sigma_n t_{n+1}\sigma_n t_{n+1} = t_{n+1}\sigma_n t_{n+1}\sigma_n.
\end{aligned}
$$

It is easy to check that the subset $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n t_{n+1} \sigma_n\}$ of $W(\tilde{C}_{n+1})$ satisfies the defining relations for $W(\tilde{C}_n)$. We may thus define the tower of $\tilde{C}$-type Coxeter groups by defining the following group homomorphism, for $n \geq 2$:

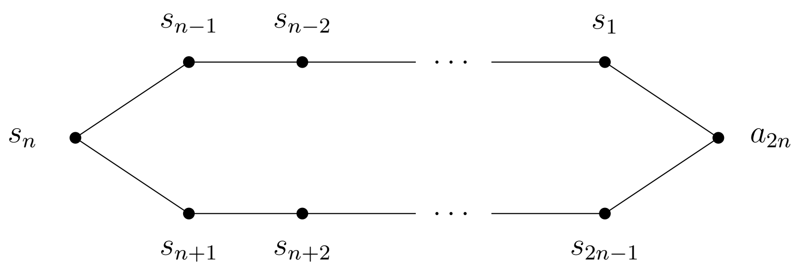
$$\begin{aligned} P_n : W(\tilde{C}_n) &\longrightarrow W(\tilde{C}_{n+1}) \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 0 \leq i \leq n-1 \\ t_n &\longmapsto \sigma_n t_{n+1} \sigma_n \end{aligned}$$

The goal of this section is to prove the faithfulness of this arrow. We will do so by embedding our $\tilde{C}$-type Coxeter groups in $\tilde{A}$-type Coxeter groups and using the faithfulness of a relevant arrow between $\tilde{A}$-type Coxeter groups, as follows. Let $W(\tilde{A}_{n-1})$ be the $\tilde{A}$-type Coxeter group with n generators, say $\{s_1, \ldots, s_{n-1}, a_n\}$, with the following Coxeter diagram:



The group $W(\tilde{C}_n)$ can be seen as the group of fixed points in $W(\tilde{A}_{2n-1})$ by some involution, so that we have an embedding of $W(\tilde{C}_n)$ in $W(\tilde{A}_{2n-1})$ given as follows (see [6, §4, §5] [14, Corollaire 3.5]; we compose here the embedding given in [6] with the Dynkin automorphism of $W(\tilde{C}_n)$) to make it more convenient for our purpose):

$$\begin{aligned} i_n : W(\tilde{C}_n) &\longrightarrow W(\tilde{A}_{2n-1}) \\ \sigma_i &\longmapsto s_{n-i} s_{n+i} \qquad \text{for } 1 \leq i \leq n-1 \\ t &\longmapsto s_n \\ t_n &\longmapsto a_{2n} \end{aligned}$$



We now recall from [1] and [3, Lemma 4.1] the following monomorphism:

$$\begin{aligned} I_n : W(\tilde{A}_{n-1}) &\longrightarrow W(\tilde{A}_n) \\ s_i &\longmapsto s_i \qquad \text{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto s_n a_{n+1} s_n \end{aligned}$$

Letting ϕ_{2n+1} be the Coxeter automorphism of $W(\tilde{A}_{2n+1})$ given by

$$s_1 \mapsto s_2 \mapsto \ldots s_{2n} \mapsto s_{2n+1} \mapsto a_{2n+2} \mapsto s_1,$$

the composition $L_n = \phi_{2n+1} I_{2n+1} I_{2n}$ is a monomorphism.

Lemma 2.1. *The following diagram commutes for any $n > 1$:*

$$\begin{tikzcd} W(\tilde{A}_{2n-1}) \arrow[r, hook, "L_n"] & W(\tilde{A}_{2n+1}) \\ W(\tilde{C}_n) \arrow[u, hook, "i_n"] \arrow[r, "P_n"'] & W(\tilde{C}_{n+1}) \arrow[u, hook, "i_{n+1}"'] \end{tikzcd}$$

Proof. For $1 \leq i \leq n-1$ we have

$$L_n i_n(\sigma_i) = L_n(s_{n-i}s_{n+i}) = s_{n-i+1}s_{n+i+1} = s_{n+1-i}s_{n+1+i} = i_{n+1}(\sigma_i) = i_{n+1}P_n(\sigma_i),$$

while for $i = 0$: $L_n i_n(t) = L_n(s_n) = s_{n+1}$ which is exactly $i_{n+1}(t)$, that is $i_{n+1}P_n(t)$.

Now consider $L_n i_n(t_n) = L_n(a_{2n})$. We have:

$$\begin{aligned} L_n(a_{2n}) &= \phi_{2n+1} I_{2n+1}(s_{2n}a_{2n+1}s_{2n}) = \phi_{2n+1}(s_{2n}s_{2n+1}a_{2n+2}s_{2n+1}s_{2n}) \\ &= s_{2n+1}a_{2n+2}s_1a_{2n+2}s_{2n+1} = s_{2n+1}s_1a_{2n+2}s_1s_{2n+1}, \end{aligned}$$

using the braid relation between s_1 and a_{2n+2}. On the other hand we have:

$$i_{n+1}P_n(t_n) = i_{n+1}(\sigma_n t_{n+1}\sigma_n) = s_{n+1-n}s_{n+1+n}a_{2n+2}s_{n+1-n}s_{n+1+n},$$

which is $s_{2n+1}s_1a_{2n+2}s_1s_{2n+1}$, and the commutation of the diagram follows. □

Corollary 2.2. *$P_n : W(\tilde{C}_n) \longrightarrow W(\tilde{C}_{n+1})$ is an injection for any $n > 1$.*

3. The tower of $\tilde{C}$-type Hecke algebras

Let for the moment K be an arbitrary commutative ring with identity; we mean by algebra in what follows K-algebra. We recall [5, Ch. IV §2 Ex. 23] that for a given Coxeter graph Γ and a corresponding Coxeter system (W, S), there is a unique algebra structure on the free K-module with basis $\{g_w \mid w \in W(\Gamma)\}$ satisfying, for a given $q \in K$:

$$\begin{aligned} g_s g_w &= g_{sw} && \text{for } s \notin \mathscr{L}(w), \\ g_s g_w &= q g_{sw} + (q-1)g_w && \text{for } s \in \mathscr{L}(w). \end{aligned}$$

We denote this algebra by $H\Gamma(q)$ and call it the the Γ-type Hecke algebra. This algebra has a presentation (*loc.cit.*) given by generators $\{g_s \mid s \in S\}$ and relations

$$\begin{aligned} g_s^2 &= q + (q-1)g_s && \text{for } s \in S, \\ (g_s g_t)^r &= (g_t g_s)^r && \text{for } s, t \in S \text{ such that } st \text{ has order } 2r, \\ (g_s g_t)^r g_s &= (g_t g_s)^r g_t && \text{for } s, t \in S \text{ such that } st \text{ has order } 2r+1. \end{aligned}$$

We assume in what follows that q is invertible in K. In this case the first defining relation above implies that g_s, for $s \in S$, is invertible with inverse

$$g_s^{-1} = \frac{1}{q}\, g_s + \frac{q-1}{q}. \tag{1}$$

We consider the $\tilde{C}_n$-type (resp. B_n-type) Hecke algebra $H\tilde{C}_n(q)$ (resp. $HB_n(q)$) corresponding to the Coxeter group $W(\tilde{C}_n)$ (resp. $W(B_n)$), for $n \geq 2$. Regarding $W(B_n)$ as a parabolic subgroup of $W(\tilde{C}_n)$ as in the previous paragraph, we view $HB_n(q)$ as the subalgebra of $H\tilde{C}_n(q)$ generated by $\{g_{\sigma_0}, g_{\sigma_1}, \dots g_{\sigma_{n-1}}\}$.

Since $W(B_n)$ is a parabolic subgroup of $W(B_{n+1})$ we can also see $HB_n(q)$ as a subalgebra of $HB_{n+1}(q)$, we thus have the following tower of Hecke algebras:

$$K \ \subset HB_2(q) \ \ \cdots \subset HB_n(q) \ \ \subset HB_{n+1}(q) \ \ \subset \cdots$$

The aim of this section is to define a similar tower of $\tilde{C}$-type Hecke algebras, despite the fact that $W(\tilde{C}_n)$ is not a parabolic subgroup of $W(\tilde{C}_{n+1})$. Let us write $\{e_{\sigma_0}, \dots, e_{\sigma_{n-1}}, e_{t_n}\}$ for the generators of $H\tilde{C}_n(q)$ and $\{g_{\sigma_0}, \dots, g_{\sigma_{n-1}}, g_{\sigma_n}, g_{t_{n+1}}\}$ for those of $H\tilde{C}_{n+1}(q)$. It is easy to check that $\{g_{\sigma_0}, \dots, g_{\sigma_{n-1}}, g_{\sigma_n}g_{t_{n+1}}g_{\sigma_n}^{-1}\}$ satisfies the defining relations for $H\tilde{C}_n(q)$, we thus get the following morphism of algebras:

$$
\begin{aligned}
R_n : H\tilde{C}_n(q) &\longrightarrow H\tilde{C}_{n+1}(q) \\
e_{\sigma_i} &\longmapsto g_{\sigma_i} \qquad \text{for } 0 \leq i \leq n-1 \\
e_{t_n} &\longmapsto g_{\sigma_n} g_{t_{n+1}} g_{\sigma_n}^{-1}.
\end{aligned} \tag{2}
$$

On the other hand, the group injection $P_n : W(\tilde{C}_n) \longrightarrow W(\tilde{C}_{n+1})$ of Corollary 2.2 extends to the group algebras, providing the following algebra monomorphism:

$$
\begin{aligned}
P_n : K[W(\tilde{C}_n)] &\longrightarrow K[W(\tilde{C}_{n+1})] \\
\sigma_i &\longmapsto \sigma_i \text{ for } 0 \leq i \leq n-1 \\
t_n &\longmapsto \sigma_n t_{n+1} \sigma_n.
\end{aligned}
$$

We let now K be the ring $\mathbf{Q}[q, q^{-1}]$ of Laurent polynomials with rational coefficients and will prove:

Proposition 3.1. *The following diagram, where M_n and M_{n+1} are the maps coming from specializing q to 1, is commutative.*

$$
\begin{tikzcd}
H\tilde{C}_n(q) \arrow[r, "R_n"] \arrow[d, "M_n"'] & H\tilde{C}_{n+1}(q) \arrow[d, "M_{n+1}"] \\
K[W(\tilde{C}_n)]) \arrow[r, hook, "P_n"'] & K[W(\tilde{C}_{n+1})]
\end{tikzcd}
$$

Proof. We will first prove the following Lemma, in which we simplify the notation by setting $R_n = R$ and $P_n = P$:

Lemma 3.2. *Let w be any element in $W(\tilde{C}_n)$. Then:*

$$
R(e_w) = A g_{P(w)} + \frac{q-1}{q^r} \sum_{x \in W(\tilde{C}_{n+1})} \lambda_x g_x,
$$

where A belongs to $q^{\mathbb{Z}}$, the λ_x are polynomials in q over $\mathbb{Q}$ and r is a non-negative integer.

Proof. Suppose $l(w) = 1$. If $w = e_{\sigma_i}$ for $0 \leq i \leq n-1$, then $R(e_w) = R(e_{\sigma_i}) = g_{\sigma_i}$, while for $w = e_{t_n}$ we use (1) and get:

$$
R(e_{t_n}) = g_{\sigma_n} g_{t_{n+1}} g_{\sigma_n}^{-1} = \frac{1}{q} g_{\sigma_n} g_{t_{n+1}} g_{\sigma_n} + \frac{1-q}{q} g_{\sigma_n} g_{t_{n+1}} = \frac{1}{q} g_{\sigma_n t_{n+1} \sigma_n} + \frac{1-q}{q} g_{\sigma_n t_{n+1}}
$$

as announced.

Now take w with $l(w) \geq 2$ and suppose that the statement is true for any element of length h where $h < l(w)$. If $w \in W(B_n)$, then $P(w) = w$ and $R(e_w) = g_w$, hence our statement holds. Otherwise w can be written as $w = ut_n v$ where $v \in W(B_n)$ and $l(w) = l(u) + l(v) + 1$ and we have:

$$R(e_w) = R(e_u)R(e_{t_n})R(e_v) = R(e_u)g_{\sigma_n}g_{t_{n+1}}g_{\sigma_n}^{-1}g_v.$$

Using (1) and the induction hypothesis for e_u we get $A \in q^{\mathbb{Z}}$, polynomials μ_y ($y \in W(\tilde{C}_{n+1})$) in $\mathbb{Q}[q]$ and $r \in \mathbb{Z}^+$ such that:

$$\begin{aligned} R(e_w) &= \frac{1}{q}\left(Ag_{P(u)} + \frac{q-1}{q^r}\sum_{y\in W(\tilde{C}_{n+1})}\mu_y g_y\right)g_{\sigma_n}g_{t_{n+1}}g_{\sigma_n}g_v \\ &\qquad + \frac{q-1}{q}\left(Ag_{P(u)} + \frac{q-1}{q^r}\sum_{y\in W(\tilde{C}_{n+1})}\mu_y g_y\right)g_{\sigma_n}g_{t_{n+1}}g_v \\ &= \frac{1}{q}Ag_{P(u)}g_{\sigma_n}g_{t_{n+1}}g_{\sigma_n}g_v + \frac{q-1}{q^{r'}}\sum_{y\in W(\tilde{C}_{n+1})}\mu'_y g_y. \end{aligned}$$

for some μ'_y ($y \in W(\tilde{C}_{n+1})$) in $\mathbb{Q}[q]$, finitely many non zero, and some $r' \in \mathbb{Z}^+$.

We can write $g_{P(u)}g_{\sigma_n} = a(q-1)g_{P(u)} + bg_{P(u)\sigma_n}$, with $(a,b) = (0,1)$ or $(a,b) = (1,q)$. Then we keep computing:

$$\begin{aligned} g_{P(u)}g_{\sigma_n}g_{t_{n+1}} &= a(q-1)g_{P(u)}g_{t_{n+1}} + bg_{P(u)\sigma_n}g_{t_{n+1}} \\ &= a(q-1)g_{P(u)}g_{t_{n+1}} + b\left(a'(q-1)g_{P(u)\sigma_n} + b'g_{P(u)\sigma_n t_{n+1}}\right) \\ &= (q-1)\left(ag_{P(u)}g_{t_{n+1}} + a'bg_{P(u)\sigma_n}\right) + bb'g_{P(u)\sigma_n t_{n+1}} \end{aligned}$$

with again $(a',b') = (0,1)$ or $(1,q)$. In the same way we multiply on the right by g_{σ_n} and we see that there exist suitable polynomials μ''_y in $\mathbb{Q}[q]$ such that:

$$g_{P(u)}g_{\sigma_n}g_{t_{n+1}}g_{\sigma_n} = (q-1)(\sum_{z\in W(\tilde{C}_{n+1})}\mu''_z g_z) + q^s g_{P(u)\sigma_n t_{n+1}\sigma_n}$$

where $0 \leq s \leq 3$. But $g_{P(u)\sigma_n t_{n+1}\sigma_n} = g_{P(u)P(t_n)} = g_{P(ut_n)}$. Moreover, since $l(w) = l(u) + l(t_n) + l(v)$ we see directly that $g_{P(u)\sigma_n t_{n+1}\sigma_n}g_v = g_{P(u\sigma_n)v} = g_{P(u\sigma_n v)} = g_{P(w)}$. Finally:

$$\begin{aligned} R(e_w) &= \frac{1}{q}A\left[(q-1)(\sum_{z\in W(\tilde{C}_{n+1})}\mu''_z g_z g_v) + q^s g_{P(w)}\right] + \frac{q-1}{q^{r'}}\sum_{y\in W(\tilde{C}_{n+1})}\mu'_y g_y \\ &= Aq^{s-1}g_{P(w)} + (q-1)\left[\frac{A}{q}(\sum_{z\in W(\tilde{C}_{n+1})}\mu''_z g_z g_v) + \frac{1}{q^{r'}}\sum_{y\in W(\tilde{C}_{n+1})}\mu'_y g_y\right]. \end{aligned}$$

The lemma follows. □

We go back to the proof of the Proposition. The diagram commutes if and only if, for each w in $W(\tilde{C}_n)$, we have:

$$P_n\Big(M_n(e_w)\Big) = M_{n+1}\Big(R_n(e_w)\Big).$$

We have $P_n\Big(M_n(e_w)\Big) = P_n(w)$ while, by specializing Lemma 3.2 at $q = 1$, we see that $M_{n+1}(R_n(e_w))$ is equal to $A(1)P_n(w)$, that is, $P_n(w)$, whence the result. □

Proposition 3.3. *Let $K = \mathbf{Q}[q, q^{-1}]$. The homomorphism of algebras*

$$R_n : H\tilde{C}_n(q) \longrightarrow H\tilde{C}_{n+1}(q)$$

defined in (2) is an injection.

Proof. We will make use of the fact that the diagram in Proposition 3.1 commutes and prove that the images of the basis elements of $H\tilde{C}_n(q)$ are linearly independent in $H\tilde{C}_{n+1}(q)$. Suppose that there exists a finite subset Z of $W(\tilde{C}_n)$ and non-zero polynomials λ_w, $w \in Z$, in $\mathbb{Q}[q, q^{-1}]$, with $\sum_{w \in Z} \lambda_w R_n(e_w) = 0$. We can as well assume that the λ_w are in $\mathbb{Q}[q]$ (by multiplying by some power of q) and that they have no common factor (by factoring out common factors if any). We do so.

Since the diagram in Proposition 3.1 commutes, we have:

$$M_{n+1}\Big(R_n(e_w)\Big) = P_n\Big(M_n(e_w)\Big) = P_n(w).$$

We now apply M_{n+1} to the dependence relation $\sum_{w \in Z} \lambda_w R_n(e_w) = 0$ to get:

$$\sum_{w \in Z} \lambda_w(1) P_n(w) = P_n(\sum_{w \in Z} \lambda_w(1) w) = 0,$$

thus $\sum_{w \in Z} \lambda_w(1) w = 0$, which implies that $\lambda_w(1) = 0$ for every $w \in Z$. This last fact means that the polynomial $(q-1)$ divides every λ_w, which contradicts our hypothesis, hence R_n is an injection. □

Remark 3.4. Whenever we know that R_n is injective, we don't need anymore to distinguish between generators of $H\tilde{C}_n(q)$ and $H\tilde{C}_{n+1}(q)$: we can denote the generators of $H\tilde{C}_n(q)$ by $g_t, g_{\sigma_1}, \ldots, g_{\sigma_{n-1}}, g_{t_n}$ and the generators of $H\tilde{C}_{n+1}(q)$ by $g_t, g_{\sigma_1}, \ldots, g_{\sigma_n}, g_{t_{n+1}}$.

4. A normal form for $\tilde{C}$-type fully commutative elements

In a given Coxeter group $W(\Gamma)$, we know that from a given reduced expression of an element w we can arrive to any other reduced expression of w only by applying braid relations [5, §1.5 Proposition 5]. Among these relations there are commutation

relations: those that correspond to generators t and s such that st has order 2.

Definition 4.1. *Elements for which one can pass from any reduced expression to any other one only by applying commutation relations are called* fully commutative elements. *We denote by $W^c(\Gamma)$, or simply W^c, the set of fully commutative elements in $W = W(\Gamma)$.*

We consider the set $W^c(B_{n+1})$ of fully commutative elements in $W(B_{n+1})$ and recall the description given by Stembridge in [17]. Using the notation there and the convention $t = \sigma_0$ we let:

$$[i, j] = \sigma_i\sigma_{i+1}\dots\sigma_j \quad \text{for } 0 \le i \le j \le n \quad \text{and} \quad [n+1, n] = 1,$$
$$[-i, j] = \sigma_i\sigma_{i-1}\dots\sigma_1 t\sigma_1\dots\sigma_{j-1}\sigma_j \quad \text{for } 1 \le i \le j \le n \quad \text{and} \quad [0, -1] = 1.$$

Theorem 4.2. [17, Theorem 5.1] *$W^c(B_{n+1})$ is the set of elements of the following form:*

$$[l_1, g_1][l_2, g_2]\dots[l_r, g_r] \tag{3}$$

with $n \ge g_1 > \dots > g_r \ge 0$ and $|l_t| \le g_t$ for $1 \le t \le r$, such that either

(1) $l_1 > \dots > l_s > l_{s+1} = \dots = l_r = 0$ for some $s \le r$, or
(2) $l_1 > \dots > l_{r-1} > -l_r > 0$.

For example: $W^c(B_2) = \{1, t, \sigma_1, t\sigma_1, \sigma_1 t, t\sigma_1 t, \sigma_1 t\sigma_1\}$. We remark that if $r > 1$, then $[l_2, g_2]\dots[l_s, g_s]$ in (3) above belongs to $W^c(B_n)$. We also notice that if σ_n appears in form (3) above, then either it appears only once and we have $n = g_1 \ne -l_1$, or it appears exactly twice and we have $n = g_1 = -l_1$.

Definition 4.3. *An element u in $W^c(B_{n+1})$ is called* extremal *if σ_n appears in a (any) reduced expression for u. In this case u can be written in one of the two following forms:*

$$\text{either} \quad [-n, n]$$
$$\text{or} \quad [l_1, n][l_2, g_2]\dots[l_s, g_s]$$

with $n > g_2 > \dots > g_s \ge 0$, $|l_t| \le g_t$ for $1 \le t \le s$, $l_1 \ne -n$ and one of the conditions (1) and (2) of Theorem 4.2.

In the group $W(\tilde{C}_{n+1})$, the only braid relation involving t_{n+1} (apart from commutation relations) is

$$t_{n+1}\sigma_n t_{n+1}\sigma_n = \sigma_n t_{n+1}\sigma_n t_{n+1}$$

where the number of occurrences of t_{n+1} is the same on both sides. It follows (recall [5, §1.5 Proposition 5]) that the number of times t_{n+1} occurs in a reduced expression of an element of $W(\tilde{C}_{n+1})$ does not depend of this reduced expression.

Definition 4.4. *Let $u \in W(\tilde{C}_{n+1})$. We define the* affine length *of u to be the number of times t_{n+1} occurs in a (any) reduced expression of u. We denote it by $L(u)$.*

Lemma 4.5. *Let w be a fully commutative element in $W(\tilde{C}_{n+1})$ with $L(w) = m \geq 2$. Fix a reduced expression of w as follows:*

$$w = u_1 t_{n+1} u_2 t_{n+1} \dots u_m t_{n+1} u_{m+1}$$

with u_i, for $1 \leq i \leq m+1$, a reduced expression of a fully commutative element in $W^c(B_{n+1})$. Then $u_2, \dots, u_m$ are extremal elements and there is a reduced expression of w of the form:

$$w = [i_1, n] t_{n+1} [i_2, n] t_{n+1} \dots [i_m, n] t_{n+1} v_{m+1} \tag{4}$$

where $v_{m+1} \in W^c(B_{n+1})$, $-n \leq i_m \leq \dots i_2 \leq i_1 \leq n+1$ and $i_2 \leq n$.

Proof. Since t_{n+1} commutes with $W(B_n)$, the fact that the expression is reduced forces u_i to be extremal for $2 \leq i \leq m$. We use form (3) for u_1 and write it as $u_1 = [l_1, n] x_1$, a reduced expression with x_1 in $W^c(B_n)$ and $-n \leq l_1 \leq n+1$. Here x_1 commutes with t_{n+1} hence, setting $i_1 = l_1$, we get a reduced expression

$$w = [i_1, n] t_{n+1} x_1 u_2 t_{n+1} \dots u_m t_{n+1} u_{m+1}.$$

Again $x_1 u_2 \in W^c(B_{n+1})$ has a reduced expression $[i_2, n] x_2$ with $-n \leq i_2 \leq n$ (since $x_1 u_2$ is extremal) and x_2 in $W(B_n)$, and this x_2 commutes with t_{n+1} and can be pushed to the right, leading to

$$w = [i_1, n] t_{n+1} [i_2, n] t_{n+1} x_2 u_3 t_{n+1} \dots u_m t_{n+1} u_{m+1}.$$

Proceeding from left to right we obtain formally form (4).

Assume $i_{j+1} \geq i_j$ for some j, $1 \leq j < m$. If $0 < |i_{j+1}| < n$, the term $\sigma_{|i_{j+1}|}$ on the right of the j-th t_{n+1} (starting from the left) can be pushed to the left until we reach the braid $\sigma_{|i_{j+1}|} \sigma_{|i_{j+1}|+1} \sigma_{|i_{j+1}|}$, a contradiction to the full commutativity. If $i_{j+1} = -n$, we actually have $i_{j+1} = i_j = -n$. If $i_{j+1} = n$, then $i_j \leq n$ and our expression contains the braid $\sigma_n t_{n+1} \sigma_n t_{n+1}$, again a contradiction. Finally if $i_{j+1} = 0$, we must have $i_j = 0$ as well because a negative i_j would produce, after pushing $\sigma_{i_{j+1}} = \sigma_0$ to the left, the braid $\sigma_1 \sigma_0 \sigma_1 \sigma_0$, a contradiction. We thus get the inequalities announced. □

Lemma 4.6. *Let w be a fully commutative element in $W(\tilde{C}_{n+1})$ with $L(w) = m \geq 2$. Write w in form (4) from Lemma 4.5. We have:*

(1) If $i_s = -n$ for some s with $1 \leq s \leq m$, then $i_j = -n$ for $2 \leq j \leq m$.

(2) If $i_s = 0$ for some s with $2 \leq s \leq m$, then $i_j = 0$ for $s \leq j \leq m$.
(3) If $-1 \geq i_s \geq -n+1$ for some s with $2 \leq s \leq m$, then $s = m$ and $v_{m+1} = 1$.
(4) If $i_s > 0$ for some s with $1 \leq s \leq m-1$, then either $i_s > |i_{s+1}|$, or $s = 1$ and $i_2 = -n$.

Proof. In case (1) the inequalities in Lemma 4.5 give the result for $j \geq s$. If $s \geq 3$ the reduced expression contains $t_{n+1}[i_{s-1}, n]t_{n+1}\sigma_n$. If i_{s-1} was not equal to $-n$, we could push to the right the leftmost term t_{n+1}, which commutes with σ_j for $j < n$, getting a reduced expression that contains $t_{n+1}\sigma_n t_{n+1}\sigma_n$, a contradiction to the full commutativity.

In case (2), if $s < m$, we know from Lemma 4.5 that $i_{s+1} \leq 0$. Our reduced expression contains $t_{n+1}[0, n]t_{n+1}\sigma_{|i_{s+1}|}$. We argue as in the proof of Lemma 4.5: a negative i_{s+1} would produce either the braid $t_{n+1}\sigma_n t_{n+1}\sigma_n$ or the braid $\sigma_{|i_{s+1}|}\sigma_{|i_{s+1}|+1}\sigma_{|i_{s+1}|}$, a contradiction.

For case (3) we observe similarly that the expression $t_{n+1}[i_s, n]t_{n+1}\sigma_j$ would produce the braid $\sigma_{j+1}\sigma_j\sigma_{j+1}$ if $0 < j < n$, the braid $t_{n+1}\sigma_n t_{n+1}\sigma_n$ if $j = n$, and the braid $\sigma_1\sigma_0\sigma_1\sigma_0$ if $j = 0$. This leaves no possibility other than the one announced.

For case (4), we observe again that $i_s \leq |i_{s+1}| < n$ would produce the braid $\sigma_{|i_{s+1}|}\sigma_{|i_{s+1}|+1}\sigma_{|i_{s+1}|}$, and $i_{s+1} = n$ would produce the braid $\sigma_n t_{n+1}\sigma_n t_{n+1}$. We are left with checking the case $i_{s+1} = -n$. If $s > 1$ this produces the braid $t_{n+1}\sigma_n t_{n+1}\sigma_n$ because the $(s-1)$-th t_{n+1} from the left can be pushed to the right until it reaches σ_n, whence the result. □

With these lemmas in hand we are ready to present the classification of fully commutative elements in $W(\tilde{C}_{n+1})$.

Theorem 4.7. *Let $w \in W^c(\tilde{C}_{n+1})$ with $L(w) \geq 2$. Then w can be written in a unique way as a reduced word of one and only one of the following two forms, for non negative integers p and k:*

First type:

$$w = [i, n]t_{n+1}([-n, n]\ t_{n+1})^k([f, n])^{-1} \tag{5}$$

with $-n \leq i \leq n+1$ and $-n \leq f \leq n+1$.

Second type:

$$\begin{aligned} &w = [i_1, n]\ t_{n+1}[i_2, n]\ t_{n+1} \dots [i_p, n]\ t_{n+1}([0, n]\ t_{n+1})^k\ w_r \quad \text{if } p > 0,\\ &w = ([0, n]\ t_{n+1})^k\ w_r \quad \text{if } p = 0, \end{aligned} \tag{6}$$

with $w_r \in W^c(B_{n+1})$ and

- *if $k > 0$: $w_r = [0, r_1][0, r_2] \dots [0, r_u]$ with $-1 \leq r_u < \dots < r_1 \leq n$;*

- *if* $p > 0$*:* $n+1 \geq i_1 > ... > i_{p-1} > |i_p| > 0$ *;*
- *if* $p > 0$ *and* $i_p < 0$*:* $k = 0$*,* $w_r = 1$ *and* $i_p \neq -n$*;*
- *if* $k = 0$ *and* $i_p > 0$*:* w_r *is of form (3) such that* $|l_1| < i_p$*.*

The affine length of w *of the first (resp. second) type is* $k+1$ *(resp.* $p+k$*) and we have* $0 \leq p \leq n+1$*.*

Now suppose that $L(w) = 1$*, then it has a reduced expression of the form:*

$$[i, n]\ t_{n+1}\ v \tag{7}$$

where

- *if* $0 < i \leq n+1$ *then* v *is of the form (3) such that for* $1 \leq j \leq r$ *either* $l_j = n-j+1$ *or* $l_j < i$*;*
- *if* $i < 0$ *then* $v = ([h, n])^{-1}$ *with* $-n \leq h \leq n+1$*;*
- *if* $i = 0$ *then either* v *is equal to* $([h, n])^{-1}$ *for* $-n \leq h \leq n+1$*, or to* $([z, n])^{-1}[0, r_1][0, r_2] \dots [0, r_m]$ *for* $-1 \leq r_m < \dots r_2 < r_1 < z \leq n+1$*.*

Conversely, every w *of the above form is in* $W^c(\tilde{C}_{n+1})$*.*

Proof. We start with an element $w \in W^c(\tilde{C}_{n+1})$ with $L(w) = m \geq 2$, written as in (4), and we discuss according to the value of i_2.

1. If $i_2 = -n$, we get from Lemma 4.6 an element of the first type: the same arguments show that any reduced expression of the rightmost term v_{m+1} must start with σ_n on the left, which forces the shape of v_{m+1}.
2. If $-(n-1) \leq i_2 < 0$ Lemma 4.6 gives directly a second type element with $p = 2$, $k = 0$ and $w_r = 1$.
3. If $i_2 = 0$ we get $i_s = 0$ for $2 \leq s \leq m$ from Lemma 4.6 and $i_1 \geq 0$ from Lemma 4.5. We thus have an element of the second type, with $p = 0$ if $i_1 = 0$, or $p = 1$ if $i_1 > 0$. Any reduced expression of the rightmost term v_{m+1} must start with σ_0 on the left, which forces the shape of $v_{m+1} = w_r$.
4. If $1 \leq i_2 \leq n$, then we must have $|i_3| < i_2$ by Lemma 4.6. We iterate this process until we find j with $i_2 > \cdots > i_j \geq 1$ and either $j = m$, hence we have an element of the second type with $p = m$ and $k = 0$, or $i_{j+1} \leq 0$, which gives an element of the second type, with $p = j$ and $k > 0$ if $i_{j+1} = 0$, or $p = j+1$, $k = 0$ and $w_r = 1$ if $i_{j+1} < 0$ (Lemma 4.6). When k is positive, w_r is as in case (3). When $k = 0$ the condition on w_r follows as in the proof of Lemma 4.6 (4).

For an element $w \in W^c(\tilde{C}_{n+1})$ of affine length $L(w) = 1$, written $w = [i, n]\ t_{n+1}\ v$ with $v \in W^c(B_{n+1})$, the arguments are similar, using form (3) for v. If $i < 0$ (resp.

$i = 0$) any reduced expression of v must start with σ_n (resp. σ_n or σ_0) on the left. If $i = n+1$, there is no further condition on v, while if $0 < i \leq n$ any reduced expression of v must start with σ_n or σ_t with $t < i$.

The fact that any element of one of these forms is fully commutative is proven by an easy induction. □

We remark that elements of the first type and elements of affine length 1 of the form $[i,n]\ t_{n+1}([h,n])^{-1}$ have a unique reduced expression. Moreover, an element of affine length at least 2 has a unique reduced expression if and only if it is of the first type. Inserting the elements of affine length 1 in the first type and second type sets would not have given us a partition of the set of those elements as we will see in the next example. This is the reason why we handle them separately.

Example 4.8. We list the elements in $W^c(\tilde{C}_2)$ of positive affine length.

- *First type elements:*

$$c\ (\sigma_1 t \sigma_1 t_2)^h\ d \qquad \textit{with} \quad \begin{cases} h \geq 1, \\ c \in \{1, t_2, \sigma_1 t_2, t\sigma_1 t_2\}, \\ d \in \{1, \sigma_1, \sigma_1 t, \sigma_1 t \sigma_1\}, \\ \textit{if } c = 1, \textit{then } h \geq 2. \end{cases}$$

- *Second type elements:*

$$a\ (t\sigma_1 t_2)^k\ b \qquad \textit{with} \quad \begin{cases} k \geq 1, \\ a \in \{1, t_2, \sigma_1 t_2\}, \\ b \in \{1, \sigma_1, t, \sigma_1 t, t\sigma_1\}, \\ \textit{if } a = 1, \textit{then } k \geq 2. \end{cases}$$

- *Elements of affine length* 1*:*

$$et_2 f \quad \textit{with} \quad \begin{cases} \textit{either } e = \sigma_1 t \sigma_1 \textit{ and } f \in \{1, \sigma_1, t\sigma_1, \sigma_1 t \sigma_1\}, \\ \textit{or } e \in \{1, \sigma_1, t\sigma_1\} \textit{ and } f \in \{1, t, t\sigma_1, t\sigma_1 t, \sigma_1, \sigma_1 t, \sigma_1 t \sigma_1\}. \end{cases}$$

Notice that if h and k were allowed to be null, then $\sigma_1 t_2 \sigma_1$ could be obtained in two different ways: $a = \sigma_1 t_2, b = \sigma_1$, $k = 0$, and $c = \sigma_1 t_2, d = \sigma_1$, $h = 0$.

Remark 4.9. In [4] the authors define and study *cyclically fully commutative* elements: elements for which a cyclic permutation of the terms of any reduced expression transforms it into a reduced expression for a fully commutative element. The normal form given in Theorem 4.7 may be used for such a study. Indeed let w be a first type element given in its normal form. Since it has a unique reduced expression, it is easy to see that w is cyclically fully commutative if and only if either $0 \leq i \leq n+1$ and $f = -(i-1)$, or $-n \leq i < 0$ and $f = -(i+1)$. In this case, after an $n-(i+1)$ (first case) or $n-(i-1)$ (second case) cyclic shift, w is transformed into $([-n,n]\ t_{n+1})^{k+1}$.

Let now w be an element of the second type given in its normal form with $k > 0$. Suppose that w is cyclically fully commutative, then $m = p$, moreover:

$$w = [i_1, n]\ t_{n+1}[i_2, n]\ t_{n+1} \dots [i_p, n]\ t_{n+1}([0, n]\ t_{n+1})^k[0, i_1 - 1][0, i_2 - 1] \dots [0, i_p - 1].$$

In this case w is transformed into $([0, n]\ t_{n+1})^{k+p}$ by a suitable cyclic shift.

5. The tower of $\tilde{C}$-type fully commutative elements

The Coxeter group $W(B_n)$ with Coxeter generators $t, \sigma_1, \cdots, \sigma_{n-1}$, is a parabolic subgroup of $W(B_{n+1})$. This is no longer the case for $W(\tilde{C}_n)$ and $W(\tilde{C}_{n+1})$ – proper parabolic subgroups of $W(\tilde{C}_{n+1})$ are finite. This is an important difficulty when dealing with the affine case. As for $W(\tilde{C}_n)$, the injection $P_n : W(\tilde{C}_n) \longrightarrow W(\tilde{C}_{n+1})$ of Corollary 2.2 is a group monomorphism that preserves the full commutativity of first type elements and elements of affine length 1 in $W^c(\tilde{C}_n)$, but does not preserve it for $t_n[0, n-1]t_n$, for example, in the set of second type fully commutative elements. We will take advantage of the normal form for fully commutative elements established in Theorem 4.7 to produce embeddings from $W^c(\tilde{C}_n)$ into $W^c(\tilde{C}_{n+1})$.

For $n > 0$, we denote by $W^c_1(\tilde{C}_n)$ the set of first type fully commutative elements in addition to fully commutative elements of affine length 1, and by $W^c_2(\tilde{C}_n)$ the set of second type fully commutative elements. We thus have the following partition:

$$W^c(\tilde{C}_n) = W^c_1(\tilde{C}_n) \bigsqcup W^c_2(\tilde{C}_n) \bigsqcup W^c(B_n).$$

Definition 5.1. *For any $w \in W^c(\tilde{C}_n)$ we define elements $I(w)$ and $J(w)$ of $W(\tilde{C}_{n+1})$ by the following expressions:*

- *if $w \in W^c_2(\tilde{C}_n)$, then $I(w)$ (resp. $J(w)$) is obtained by substituting $\sigma_n t_{n+1}$ (resp. $t_{n+1}\sigma_n$) to t_n in the normal form (6) for w;*
- *if $w \in W^c_1(\tilde{C}_n)$, then $I(w) = J(w)$ is obtained by substituting $\sigma_n t_{n+1} \sigma_n$ to t_n in the normal form (5) or (7) for w;*
- *if $w \in W^c(B_n)$, then $I(w) = J(w) = w$.*

Theorem 5.2. *For any $w \in W^c(\tilde{C}_n)$, the expressions for $I(w)$ and $J(w)$ in Definition 5.1 are reduced and they are reduced expressions for fully commutative elements in $W(\tilde{C}_{n+1})$. The maps thus defined:*

$$I, J : W^c(\tilde{C}_n) \longrightarrow W^c(\tilde{C}_{n+1})$$

are injective, preserve the affine length and satisfy

$$l(I(w)) = l(J(w)) = l(w) + L(w) \quad \text{for } w \in W^c_2(\tilde{C}_n),$$
$$l(I(w)) = l(J(w)) = l(w) + 2L(w) \quad \text{for } w \in W^c_1(\tilde{C}_n).$$

The injections I and J map first type (resp. second type) elements to first type (resp. second type) elements and their images intersect exactly on $I(W_1^c(\tilde{C}_n) \bigsqcup W^c(B_n))$.

Proof. We have for $-(n-1) \leq i, f \leq n$:

$$I([i, n-1]t_n([-(n-1), n-1]\ t_n)^k([f, n-1])^{-1}) = [i, n]t_{n+1}([-n, n]\ t_{n+1})^k([f, n])^{-1}$$

hence if w in $W^c(\tilde{C}_n)$ is a first type element written in form (5), then the expression $I(w) = J(w)$ is the normal form (5) of a first type element in $W^c(\tilde{C}_{n+1})$.

Similarly if $w = [i, n-1]t_n v$ is fully commutative of affine length 1 written in form (7), we have:

$$I(w) = [i, n-1]\sigma_n t_{n+1}\sigma_n v = [i, n]t_{n+1}(\sigma_n v)$$

which is the normal form (7) of an element of affine length 1 in $W^c(\tilde{C}_{n+1})$.

Now let w be a second type element written in form (6), we see directly that

$$\begin{aligned} I(w) &= I([i_1, n-1]\ t_n[i_2, n-1]\ t_n \dots [i_p, n-1]\ t_n([0, n-1]\ t_n)^k w_r) \\ &= [i_1, n]\ t_{n+1}[i_2, n]\ t_{n+1} \dots [i_p, n]\ t_{n+1}([0, n]\ t_{n+1})^k w_r, \end{aligned}$$

the normal form (6) of a second type element in $W^c(\tilde{C}_{n+1})$. We now compute $J(w)$, recalling that t_{n+1} commutes with σ_i for $0 \leq i < n$:

$$\begin{aligned} J(w) &= J([i_1, n-1]\ t_n[i_2, n-1]\ t_n \dots [i_p, n-1]\ t_n([0, n-1]\ t_n)^k w_r) \\ &= [i_1, n-1]\ t_{n+1}\sigma_n[i_2, n-1]\ t_{n+1}\sigma_n \dots [i_p, n-1]\ t_{n+1}\sigma_n([0, n-1]\ t_{n+1}\sigma_n)^k w_r \\ &= \ t_{n+1}[i_1, n-1]\sigma_n\ t_{n+1}[i_2, n-1]\sigma_n \dots\ t_{n+1}[i_p, n-1]\sigma_n(\ t_{n+1}[0, n-1]\sigma_n)^k w_r \\ &= \ t_{n+1}[i_1, n]\ t_{n+1}[i_2, n] \dots\ t_{n+1}[i_p, n](\ t_{n+1}[0, n])^k w_r \end{aligned}$$

If $k = 0$, we check that $w_r' = [i_p, n]w_r$ satisfies the condition on the rightmost term in form (6). If $k > 0$:

$$J(w) = \ t_{n+1}[i_1, n]\ t_{n+1}[i_2, n] \dots\ t_{n+1}[i_p, n]t_{n+1}(\ [0, n]t_{n+1})^{k-1}[0, n]w_r,$$

where again $w_r' = [0, n]w_r$ satisfies the condition Theorem 4.7. Hence $J(w)$ is the normal form of a second type element in $W^c(\tilde{C}_{n+1})$. We notice that the leftmost term in the expression of $J(w)$ is t_{n+1} whereas no reduced expression of $I(w)$ can have t_{n+1} as its leftmost term (since the expression $t_{n+1}I(w)$ is also a normal form of the second type), therefore the images $I(W_2^c(\tilde{C}_n))$ and $J(W_2^c(\tilde{C}_n))$ are disjoint.

I and J clearly preserve the affine length. The fact that the substitution process on $W_2^c(\tilde{C}_n)$ (resp. on $W_1^c(\tilde{C}_n)$) adds to the original length the number of occurrences (resp. the double of the number of occurrences) of t_n, i.e. the affine length or its

double, is clear. The injectivity of both maps results from the uniqueness of the normal form. $\square$

We remark that the injections I and J on $W_1^c(\tilde{C}_n)\bigsqcup W^c(B_n)$ are but the restriction of R_n. Actually I and J may be defined on all $W(\tilde{C}_n)$, but as we don't need this, we won't examine it further.

6. The tower of $\tilde{C}$-type Temperley-Lieb algebras

Let K be an integral domain of characteristic 0 and let q be an invertible element in K. Let Γ be a Coxeter graph with associated Coxeter system (W, S) and Hecke algebra $H\Gamma(q)$. Following Graham [10, Definition 6.1], we define the Γ-type Temperley-Lieb algebra $TL\Gamma(q)$ to be the quotient of the Hecke algebra $H\Gamma(q)$ by the two-sided ideal generated by the elements $L_{s,t} = \sum_{w\in <s,t>} g_w$, where s and t are non commuting elements in S such that st has finite order. For w in W we denote by T_w the image of $g_w \in H\Gamma(q)$ under the canonical surjection from $H\Gamma(q)$ onto $TL\Gamma(q)$. The set $\{T_w \mid w \in W^c(\Gamma)\}$ forms a K-basis for $TL\Gamma(q)$ [10, Theorem 6.2].

For x, y in a given ring with identity, we define:

$$V(x,y) = xyx + xy + yx + x + y + 1,$$
$$Z(x,y) = xyxy + xyx + yxy + xy + yx + x + y + 1.$$

For $n \geq 1$, the $\tilde{C}$-type Temperley-Lieb algebra with $n+2$ generators $TL\tilde{C}_{n+1}(q)$ is given by the set of generators $\left\{T_t, T_{\sigma_1}, \dots T_{\sigma_n}, T_{t_{n+1}}\right\}$, with the defining relations (with our convention $t = \sigma_0$):

$$(8) \qquad \begin{cases} T_{\sigma_i}T_{\sigma_j} = T_{\sigma_j}T_{\sigma_i} \text{ for } 0 \leq i,j \leq n \text{ and } |i-j| \geq 2, \\ T_{\sigma_i}T_{t_{n+1}} = T_{t_{n+1}}T_{\sigma_i} \text{ for } 0 \leq i \leq n-1, \\ T_{\sigma_i}T_{\sigma_{i+1}}T_{\sigma_i} = T_{\sigma_{i+1}}T_{\sigma_i}T_{\sigma_{i+1}} \text{ for } 1 \leq i \leq n-1, \\ T_tT_{\sigma_1}T_tT_{\sigma_1} = T_{\sigma_1}T_tT_{\sigma_1}T_t, \\ T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}T_{\sigma_n} = T_{\sigma_n}T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}, \\ T_{t_{n+1}}^2 = (q-1)T_{t_{n+1}} + q \text{ and } T_{\sigma_i}^2 = (q-1)T_{\sigma_i} + q \text{ for } 0 \leq i \leq n, \\ V(T_{\sigma_i}, T_{\sigma_{i+1}}) = 0 \text{ for } 1 \leq i \leq n-1, \\ Z(T_{\sigma_1}, T_t) = Z(T_{\sigma_n}, T_{t_{n+1}}) = 0. \end{cases}$$

We set $TL\tilde{C}_1(q) = K$. In the following we denote by h_w, $w \in W^c(\tilde{C}_n)$, the basis elements of $TL\tilde{C}_n(q)$ to distinguish them from those of $TL\tilde{C}_{n+1}(q)$.

Lemma 6.1. *The morphism of algebras $R_n : H\tilde{C}_n(q) \longrightarrow H\tilde{C}_{n+1}(q)$ defined in (2) induces the following morphism of algebras, which we also denote by R_n:*

$$\begin{aligned} R_n : TL\tilde{C}_n(q) &\longrightarrow TL\tilde{C}_{n+1}(q) \\ h_{\sigma_i} &\longmapsto T_{\sigma_i} \quad \text{for } 0 \le i \le n-1 \\ h_{t_n} &\longmapsto T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n}^{-1}. \end{aligned}$$

The restriction of R_n to $TLB_n(q)$ is an injective morphism into $TLB_{n+1}(q)$ and satisfies $R_n(h_w) = g_{I(w)} = g_{J(w)}$ for $w \in W^c(B_n)$.

Proof. The lemma follows after noticing that

$$Z(R_n(h_{\sigma_{n-1}}), R_n(h_{t_n})) = (T_{\sigma_n} T_{\sigma_{n-1}}) Z(T_{\sigma_n}, T_{t_{n+1}}) (T_{\sigma_n} T_{\sigma_{n-1}})^{-1}.$$

□

The aim of this section is to show, using the normal form of Theorem 4.7, that the morphism R_n is an injection. We set $p = 1/q$. We will use repeatedly

$$R_n(h_{t_n}) = pT_{\sigma_n t_{n+1} \sigma_n} + (p-1) T_{\sigma_n t_{n+1}}, \tag{9}$$

as well as the following consequences of the defining relations (8):

(i) In $TL\tilde{C}_{n+1}(q)$, a product $T_w T_y$, $w, y \in W^c(\tilde{C}_{n+1})$, is a linear combination of terms T_z, $z \in W^c(\tilde{C}_{n+1})$, with $L(z) \le L(w) + L(y)$ and $l(z) \le l(w) + l(y)$.

(ii) When a braid $T_{\sigma_i} T_{\sigma_{i+1}} T_{\sigma_i}$ appears in a computation, the use of $V(T_{\sigma_i}, T_{\sigma_{i+1}}) = 0$ replaces it by a sum of terms T_z with $l(z) = 2$, 1 or 0, hence the length decreases.

(iii) When a braid $T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n}$ occurs, the use of $Z(T_{\sigma_n}, T_{t_{n+1}}) = 0$ replaces it by a sum of terms T_z with $l(z) \le 3$ in which only one has affine length 2, namely $T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$. The other terms have affine length 1 or 0 and will be ignored since we will be interested in those terms with maximal affine length.

Lemma 6.2. *If $w \in W_1^c(\tilde{C}_n) \bigsqcup W^c(B_n)$ we have:*

$$R_n(h_w) = p^{L(w)} T_{I(w)} + \sum_{\substack{L(x) \le L(w) \\ l(x) < l(I(w))}} \alpha_x T_x \qquad (\alpha_x \in K).$$

Proof. We prove the statement by induction on the affine length $L(w)$ of w, recalling that for $w \in W^c(B_n)$ we have $R_n(h_w) = T_{I(w)}$, which implies the assertion for affine length 0. We now assume that the property holds for any u of affine length at most k.

Let w in $W_1^c(\tilde{C}_n)$ have affine length $L(w) = k+1$ and write w in its normal form given by Theorem 4.7 so that $w = ut_nv$ where u is an element in $\in W_1^c(\tilde{C}_n) \bigsqcup W^c(B_n)$ written in normal form, $L(u) = k$, $v \in W^c(B_n)$ and $l(w) = l(u) + l(v) + 1$. We have:

$$I(w) = J(w) = I(u)\sigma_n t_{n+1}\sigma_n I(v).$$

Since R_n is an homomorphism of algebras we have, using (9):

$$R_n(h_w) = R_n(h_u)R_n(h_{t_n})R_n(h_v) = R_n(h_u)\ [\ pT_{\sigma_n t_{n+1}\sigma_n} + (p-1)T_{\sigma_n t_{n+1}}]\ T_{I(v)}.$$

By the induction hypothesis, $R_n(h_u)$ is a linear combination of terms T_z with $L(z) \leq k$, and has a unique term of maximal length (that is, a term T_z where z has maximal length), which is $p^{L(u)}T_{I(u)}$. Recalling (i) above, we deduce that $R_n(h_w)$ is a linear combination of terms T_y with $L(y) \leq k+1$, and has a unique term of maximal length which is $p^{L(u)}T_{I(u)}pT_{\sigma_n t_{n+1}\sigma_n}T_{I(v)} = p^{L(w)}T_{I(w)}$. $\square$

Proposition 6.3. *Let w be in $W_2^c(\tilde{C}_n)$, then for some $\alpha_x, \beta_y \in K$ we have*

$$R_n(h_w) = (-1)^{L(w)}T_{I(w)} + (-p)^{L(w)}T_{J(w)} + \sum_{\substack{L(y)=L(w)\\ l(y)<l(I(w))}} \beta_y T_y + \sum_{L(x)<L(I(w))} \alpha_x T_x.$$

Proof. We use the normal form (6) of Theorem 4.7 for $w \in W_2^c(\tilde{C}_n)$ and we remark first that it is enough to prove our assertion for $w_r = 1$ and $i_1 = n$. Indeed, the normal form for w is $w = [i_1, n-1]uw_r$ where u is an element of $W_2^c(\tilde{C}_n)$ having same affine length as w, whose normal form begins and ends with t_n. Then, assuming our result holds for u, we have:

$$\begin{aligned} R_n(h_w) =& R_n(h_{[i_1,n-1]})R_n(h_u)R_n(h_{w_r}) \\ = T_{[i_1,n-1]}&((-1)^{L(u)}T_{I(u)} + (-p)^{L(u)}T_{J(u)} + \sum_{\substack{L(y)=L(u)\\ l(y)<l(I(u))}} \beta_y T_y + \sum_{L(x)<L(I(u))} \alpha_x T_x)T_{I(w_r)}, \end{aligned}$$

hence our result since $I(w) = [i_1, n-1]I(u)I(w_r)$ and $J(w) = [i_1, n-1]J(u)J(w_r)$, both expressions reduced and fully commutative, and the remaining terms will have either the same affine length as $I(w)$ but a smaller Coxeter length, or a smaller affine length.

We work by induction on the affine length and remark once and for all that the development of $R_n(h_w)$ will contain only terms T_z with $L(z) \leq L(w)$ (see (i) above). To prove our claim, we will then focus on terms of affine length $L(w)$. We will actually prove by induction the following more precise statement:

Let w be in $W_2^c(\tilde{C}_n)$, whose normal form (6) begins and ends with t_n. Then for some $\alpha_x, \beta_y \in K$ we have:

$$
\begin{aligned}
R_n(h_w) = (-1)^{L(w)} T_{I(w)} + (-p)^{L(w)} T_{J(w)} \\
+ T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} \Big(\sum_{\substack{L(y)=L(w)-2 \\ l(y)<l(I(w))-3}} \beta_y T_y \Big) + \sum_{L(x)<L(I(w))} \alpha_x T_x .
\end{aligned}
\tag{10}
$$

We start with an element $w = t_n[i, n-1]t_n$ for $-(n-1) < i \le n-1$. Using (9) we see that $R_n(h_w) = R_n(h_{t_n}) R_n(h_{[i,n-1]}) R_n(h_{t_n})$ is a linear combination of elements of the basis of $TL\tilde{C}_{n+1}(q)$ with affine length at most 2, appearing in the following products:

$$
\begin{array}{ccccc}
[1]\ (p-1)\ T_{\sigma_n t_{n+1}} & \searrow & & \swarrow & [1']\ (p-1)\ T_{\sigma_n t_{n+1}} \\
 & & T_{[i,n-1]} & & \\
[2]\ p\ T_{\sigma_n t_{n+1} \sigma_n} & \nearrow & & \nwarrow & [2']\ p\ T_{\sigma_n t_{n+1} \sigma_n}
\end{array}
$$

The product associated to [1] and [1'] is $(p-1)^2 T_{I(w)}$.

Developping the one associated to [1] and [2'], since $T_{t_{n+1}}$ commutes with $T_{[i,n-1]}$, we find the braid

(11)
$$T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n} = -T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} - T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n} - T_{t_{n+1}} T_{\sigma_n} - T_{\sigma_n} T_{t_{n+1}} - T_{t_{n+1}} - T_{\sigma_n} - 1$$

so, recalling (iii) above, we get the term $-p(p-1)T_{I(w)}$ and terms of affine length at most 1.

Developping the product corresponding to [2] and [1'] we find the braid

$$T_{\sigma_n} T_{\sigma_{n-1}} T_{\sigma_n} = -T_{\sigma_n} T_{\sigma_{n-1}} - T_{\sigma_{n-1}} T_{\sigma_n} - T_{\sigma_n} - T_{\sigma_{n-1}} - 1. \tag{12}$$

The first term here leads to the braid $T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$, hence to only one term of affine length 2, namely $p(p-1) T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{[i,n-1]}$ which has length $l(I(w)) - 1$. The second leads to $-p(p-1)T_{I(w)}$. The third $-T_{\sigma_n}$ leads to the braid (11) again, of which we keep only the first term, hence a $p(p-1) T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{[i,n-2]}$ which has length $l(I(w)) - 2$. The fourth and the fifth terms lead to non reduced expressions containing $T^2_{t_{n+1}}$, hence to terms of affine length strictly less than $L(w)$.

Similarly, developping the product with [2] and [2'], we find the braid $T_{\sigma_n} T_{\sigma_{n-1}} T_{\sigma_n}$. The first term in (12) leads to the braid $T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$ of which we keep the term $-T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$, leading to a final term $p^2 T_{J(w)}$. The second term in (12) leads to $T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{\sigma_n}$ of which we keep as above $-T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$, leading to $p^2 T_{I(w)}$. The third $-T_{\sigma_n}$ leads to the braid (11), of which we keep the first term, hence a

$p^2 T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{[i,n-2]} T_{\sigma_n}$ where $T_{[i,n-2]} T_{\sigma_n}$ has length at most $l(I(w)) - 4$ (we could reduce this term again with (11) but this is the form suited for our purpose). With the fourth and fifth terms we get as before a $T^2_{t_{n+1}}$, decreasing the affine length.

Summing up we get our two elements:

$$((p-1)^2 + p^2 + 2p(1-p))T_{I(w)} = T_{I(w)} \quad \text{and} \quad p^2 T_{J(w)},$$

plus a product of $T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$ by terms of affine length 0 and Coxeter length at most $l(I(w)) - 4$, plus terms of smaller affine length, as announced.

Now that we have detailed the case of affine length 2, we will be quicker with the induction step. Indeed take w in $W_2^c(\tilde{C}_n)$, in its normal form (6) beginning and ending with t_n, of affine length at least 3, and write it $w = t_n[i, n-1]u$ where $-(n-1) < i \leq n-1$ and u satisfies the same conditions except that $L(u) \geq 2$. Assuming that our statement holds for u we write:

$$\begin{aligned} R_n(h_w) &= R_n(h_{t_n}) R_n(h_{[i,n-1]}) R_n(h_u) \\ &= (pT_{\sigma_n t_{n+1} \sigma_n} + (p-1)T_{\sigma_n t_{n+1}})\ T_{[i,n-1]}\ [(-1)^{L(u)} T_{I(u)} \\ &+ (-p)^{L(u)} T_{J(u)} + T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}\ (\sum_{\substack{L(y)=L(u)-2 \\ l(y)<l(I(u))-3}} \beta_y T_y) + \sum_{L(x)<L(I(u))} \alpha_x T_x] \end{aligned} \tag{13}$$

and examine the terms of this product, ignoring the sum over x with $L(x) < L(I(u))$ that will anyway lead to terms T_z with $L(z) < L(I(u))$ (rule (i)). We draw as before:

$$\begin{array}{ccccc} [1]\ (p-1)\ T_{\sigma_n t_{n+1}} & & & & [1']\ (-1)^{L(u)} T_{I(u)} \\ & & T_{[i,n-1]} & & [2']\ (-p)^{L(u)} T_{J(u)} \\ [2]\ p\ T_{\sigma_n t_{n+1} \sigma_n} & & & & [3']\ T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} \end{array}$$

[1] and [3']: we have a $T^2_{t_{n+1}}$ so the affine length drops.

[2] and [3']: we have the braid (11) which we replace by $-T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$, leading to products of $T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$ with terms coming from $T_{[i,n-1]} T_{\sigma_n} T_{t_{n+1}} T_y$ hence terms T_z with $L(z) \leq L(w) - 2$ and $l(z) < l(I(u)) - 3 + l([i, n-1]\sigma_n t_{n+1}) = l(I(w) - 3$.

[1] and [2']: $J(u)$ is fully commutative with a reduced expression starting with t_{n+1}, hence $T_{J(u)} = T_{t_{n+1}} T_v$ and we get a $T^2_{t_{n+1}}$ dropping the affine length.

[2] and [2']: we write again $T_{J(u)} = T_{t_{n+1}} T_v$ and get the braid (11) which we replace by $-T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}}$. We thus get:

$$(-p)(-p)^{L(u)} T_{t_{n+1}} T_{\sigma_n} T_{t_{n+1}} T_{[i,n-1]} T_v = (-p)^{L(w)} T_{J(w)}.$$

[1] and [1']: this gives directly $(p-1)(-1)^{L(u)}T_{I(w)}$.

[2] and [1']: $I(u)$ is fully commutative with a reduced expression starting with $\sigma_n t_{n+1}$, we write then $T_{I(u)} = T_{\sigma_n}T_{t_{n+1}}T_s$ and get the braid (12). The first term $-T_{\sigma_n}T_{\sigma_{n-1}}$ in (12), leads to (11) which we replace by $-T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}$, itself multiplied on the right by $T_{[i,n-1]}T_s$ that develops as a sum of terms T_z of affine length at most $L(I(u))-1 = L(I(w))-2$ as expected. The second term $-T_{\sigma_{n-1}}T_{\sigma_n}$ in (12) provides the term $(-p)(-1)^{L(u)}T_{I(w)}$ that, added with the one from ([1] and [1']), gives us the $(-1)^{L(w)}T_{I(w)}$ that we need. The third term $-T_{\sigma_n}$ in (12) again leads to (11) hence a term starting with $T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}$. The fourth and fifth will provide a $T^2_{t_{n+1}}$ dropping the affine length. ☐

Theorem 6.4. *The tower of affine Temperley-Lieb algebras*

$$TL\tilde{C}_1(q) \xrightarrow{R_1} TL\tilde{C}_2(q) \xrightarrow{R_2} TL\tilde{C}_3(q) \longrightarrow \cdots TL\tilde{C}_n(q) \xrightarrow{R_n} TL\tilde{C}_{n+1}(q) \longrightarrow \cdots$$

is a tower of faithful arrows.

Proof. We need to show that R_n is an injective homomorphism of algebras. A basis for $TL\tilde{C}_n(q)$ is given by the elements h_w where w runs over $W^c(\tilde{C}_n)$. Assume there are non trivial dependence relations between the images of these basis elements. Pick one such relation, say

$$\sum_w \lambda_w R_n(h_w) = 0, \tag{14}$$

and let $m = \max\{L(w) \mid w \in W^c(\tilde{C}_n) \text{ and } \lambda_w \neq 0\}$. If $m = 0$, all elements w such that $\lambda_w \neq 0$ belong to $W^c(B_n)$, contradicting the injectivity of the restriction of R_n to $TLB_n(q)$ (Lemma 6.1). So m is positive.

We know from Lemma 6.2 and Proposition 6.3 that for $w \in W^c(\tilde{C}_n)$, $R_n(h_w)$ is a linear combination of terms T_z where z has affine length at most $L(w)$. Therefore the terms T_z where z has maximal affine length in the development of (14) are exactly the terms of maximal affine length in the development of $\sum_w \lambda_w R_n(h_w)$ where w runs over the set of fully commutative elements of affine length m in $W(\tilde{C}_n)$. Among them, Lemma 6.2 and Proposition 6.3 give us the terms of maximal Coxeter length, hence the terms of maximal Coxeter length among the terms of maximal affine length m in the development of (14) are the terms of maximal Coxeter length in the sum:

$$\sum_{\substack{x \in W_1^c(\tilde{C}_n)\\ L(x)=m \\ \alpha_x \neq 0}} \alpha_x T_{I(x)} \;+\; \sum_{\substack{x \in W_2^c(\tilde{C}_n)\\ L(x)=m \\ \alpha_x \neq 0}} \alpha_x((-1)^m T_{I(x)} + (-p)^m T_{J(x)}) \tag{15}$$

The elements T_d for $d \in W^c(\tilde{C}_{n+1})$ form a basis of $\widehat{TL}_{n+1}(q)$. Moreover I and J are injective, map $W_i^c(\tilde{C}_n)$ to $W_i^c(\tilde{C}_{n+1})$ for $i = 1, 2$, and $I(W_2^c(\tilde{C}_n))$ and $J(W_2^c(\tilde{C}_n))$ are disjoint (Theorem 5.2). Therefore we see that all the coefficients α_x for $L(x) = m$ and $l(I(x)$ maximal in (15) must be 0, a contradiction. $\square$

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